Introducing mathematical notation

Standard notation is used in mathematics to simplify the writing of mathematical expressions. This notation is based on the use of symbols that you will already recognise including: = (equal), + (addition), – (subtraction), × (multiplication), and ÷ (division). Other symbols that you may not be so familiar with include < (less than), > (greater than), ∝ (proportional to) and √ (square root). You need to understand what each of these symbols means and how they are used so we shall take a brief look at those with which you might not already be familiar.

Indices

The number 4 is the same as 2 × 2, that is, 2 multiplied by itself. We can write (2 × 2) as 2^2. In words, we would call this 'two raised to the power two' or simply 'two squared'. Thus:

$2 \times 2 = \mathbf{2^2}$

By similar reasoning we can say that:

$2 \times 2 \times 2 = \mathbf{2^3}$ and $2 \times 2 \times 2 \times 2 = \mathbf{2^4}$

In these examples, the number that we have used (i.e., 2) is known as the *base* whilst the number that we have raised it to is known as an *index*. Thus, 2^4 is called 'two to the power of four', and it consists of a base of 2 and an index of 4. Similarly, 5^3 is called 'five to the power of 3' and has a base of 5 and an index of 3. Special names are used when the indices are 2 and 3, these being called 'squared' and 'cubed', respectively. Thus 7^2 is called 'seven squared' and 9^3 is called 'nine cubed'. When no index is shown, the power is 1, i.e. 2^1 means 2. Also, note that *any* number raised to the power 0, is 1. Hence, $2^0 = 1$, $3^0 = 1$, $4^0 = 1$, and so on.

Example 1

Find the value of $2^5 + 3^3$

Now $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ and $3^3 = 3 \times 3 \times 3 = 27$

So $2^5 + 3^3 = 32 + 27 = \mathbf{59}$

Example 2

Find the value of $10^2 - 5^3$

Now $10^2 = 10 \times 10 = 100$ and $5^3 = 5 \times 5 \times 5 = 125$

So $10^2 - 5^3 = 100 - 125 =$ **-25**

Reciprocals

The *reciprocal* of a number is when the index is –1 and its value is given by 1 divided by the base. Thus the reciprocal of 2 is 2^{-1} and its value is ½ or 0.5. Similarly, the reciprocal of 4 is 4^{-1} which means ¼ or 0.25.

Example 3

Find the value of $3^2 + 2^{-1}$

Now $3^2 = 3 \times 3 = 9$ and $2^{-1} =$ ½ or 0.5

So $3^2 + 2^{-1} = 9 + 0.5 =$ **9.5**

Example 4

Find the value of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

Now $\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, and $\frac{1}{8} = 0.125$

So $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.5 + 0.25 + 0.125 = \mathbf{0.875}$

Negative indices

We have already said that the reciprocal of a number is the same as that number raised to the power –1. If the reciprocal happens to be a number raised to a power other than 1 then this is the same as the number raised to the same but *negative* power. This is probably sounding a lot more complex than it really is so here are a few examples:

$\frac{1}{2} = 2^{-1}$, $\frac{1}{2^2} = 2^{-2}$ and $\frac{1}{2^3} = 2^{-3}$

Example 5

Find the value of 2^{-3}

Now $2^{-3} = \frac{1}{2^3} = \frac{1}{2\times2\times2} = \frac{1}{8} = \mathbf{0.125}$

Square roots

The *square root* of a number is when the index is ½. The square root of 2 is written as $2^{½}$ or √2. The value of a square root is the value of the base which when multiplied by itself gives the number. Since 3 × 3 = 9, then √9 = 3. However, (–3) × (–3) = 9, so we have a second possibility, i.e. √9 = ±3. There are always two answers when finding the square root of a number and we can indicate this is by placing a ± sign in front of the result meaning 'plus or minus'. Thus:

$4^{½} = \sqrt{4} = \pm 2$

and

$9^{½} = \sqrt{9} = \pm 3$.

Example 6

Find the value of $\sqrt{25}$

Now 25 = 5 × 5 (or –5 × –5)

So $\sqrt{25} = \mathbf{\pm 5}$

Example 7

Find the value of $\sqrt{100} + \sqrt{4}$

Now $\sqrt{100} = \mathbf{\pm 10}$ and $\sqrt{4} = \mathbf{\pm 2}$

So $\sqrt{100} + \sqrt{4} = \pm 10 \pm 2$

So we now have *four* potential answers;

$+10 + 2 =$ **12**

$+10 - 2 =$ **8**

$-10 + 2 =$ **–8**

$-10 - 2 =$ **–12**

Example 8

Find the value of $\dfrac{64^{\frac{1}{2}}}{16^{\frac{1}{2}}}$

Now $64^{\frac{1}{2}} = \sqrt{64} = 8$ and $16^{\frac{1}{2}} = \sqrt{16} = 4$

So $\dfrac{64^{\frac{1}{2}}}{16^{\frac{1}{2}}} = \dfrac{8}{4} = \mathbf{2}$

Electronic calculators

You will find that an electronic calculator can be extremely useful when it comes to solving engineering problems. You will need to ensure that the calculator has a full range of mathematical functions (+, –, ÷, ×, √, etc) as well the ability to use engineering notation (see later). Such calculators are often referred to as *scientific calculators*. The problems that follow all require the use of a calculator (see separate handout).

Problem 1

Simplify each of the following expressions:

1. 3.17^{2}

2. 0.45^{-1}

3. $2.5^{2} + 4.2^{-1}$

4. $9.1^3 - 0.4^{-1}$

5. $\frac{1}{1.25} + \frac{1}{4.5} + \frac{1}{8.2}$

6. $0.245^{-1} - 0.55^{-2}$

7. $\sqrt{1255}$

8. $\sqrt{19.5} + 2.2^2$

9. $\frac{6.75^2}{2.5^2}$

10. $\frac{226^{\frac{1}{2}}}{3.2^2}$

Problems 2

Which of the following mathematical statements are correct?

1. $3^4 = 12$

2. $0.5^{-1} = 2$

3. $\frac{1}{50} = 0.02$

4. $3^3 - 2^{-1} = 8.5$

5. $\frac{1}{5} + \frac{1}{4} = \frac{1}{9}$

6. $0.2^{-1} + 2^{-1} = 5.5$

7. $\sqrt{10000} = 10^2$

8. $\sqrt{0.4} = 0.2$

9. $\frac{10^2}{10} = 10$

10. $2^0 = 2$

Variables and constants

Unfortunately, we don't always know the value of a particular quantity that we need to use in a calculation. In some cases, the value might actually change, in which case we refer to it as a *variable*. In other cases, the value might be fixed but we might prefer not to actually quote its value. In this case we refer to the value as a *constant*.

In either case, we use a *symbol* to represent the quantity. The symbol itself (often a single letter) is a form of shorthand notation. For example, in the case of the voltage produced by a battery we would probably use v to represent *voltage* whereas, in the case of the time taken to travel a certain distance, we might use t.

An example of a *variable* quantity is the outside temperature. On a hot summer's afternoon, the temperature may well exceed 30°C whilst, on a cold winter's morning it might be as little as −4°C. An example of a *constant* quantity might be the temperature at which water freezes and becomes ice, i.e. 0°C.

Let's take a simple example. The weight of an empty lorry (known as its *unladen weight*) might be 2 tonnes. If it carries load of 4 tonnes its total weight (known as its *laden weight*) will be (2 + 4 = 6 tonnes). The lorry's unladen weight is a *constant* and the load weight (which might change for every trip) is a *variable*. We can express the relationship between the unladen weight, load weight, and total weight using a simple formula, as follows:

Total weight = unladen weight + load weight

$W_t = W_u + W_l$

where W_t represents the total weight, W_u represents the unladen weight, and W_l represents the load weight.

The formula can be quite useful. For example, suppose the lorry manufacturer has specified a maximum total weight of 12.5 tonnes. We might want to check that we don't exceed this value. We can calculate the maximum load weight by re-arranging the formula as follows:

$W_l = W_t - W_u$

where W_t = 12.5 tonnes and W_u = 2 tonnes.

From which:

$W_l = 12.5 - 2 = 10.5$ tonnes

Hence the maximum load that the lorry can carry (without exceeding the manufacturer’s specification) is 10.5 tonnes.

Proportionality

In engineering applications, when one quantity changes it normally affects several other quantities. For example, if the engine speed of a car increases its road speed will invariably also increase. To put this in a mathematical way we can say that “road speed is *directly proportional* to engine speed”.

Using mathematical notation and symbols to represent the quantities, we would write this as follows:

$$v \propto N$$

where v represents road speed and N represents the engine speed.

In some cases, an increase in one quantity might produce a *reduction* in another quantity. For example, if the road speed of a car increases the time taken for it to travel a measured distance will decrease. To put this in a mathematical way we would say that “time taken to travel a measured distance is *inversely proportional* road speed”.

Using mathematical notation and symbols to represent the quantities, we would write this as follows:

$$t \propto \frac{1}{v}$$

where t represents the time taken and v represents the road speed.

Example 9

The current in an electric circuit is directly proportional to the voltage applied to it and inversely proportional to the resistance of the circuit. Using I to represent current, V to represent voltage, and R to represent resistance we can say that:

$I \propto V$ (current, I, is proportional to voltage, V)

and

$I \propto \frac{1}{R}$ (current, *I*, is inversely proportional to resistance, *V*)

We can combine these two relationships to obtain an *equation* which involves all three variables, *I*, *V* and *R*:

$$I = \frac{V}{R}$$

Example 10

The power delivered to a loudspeaker is proportional to the square of the voltage applied to it and inversely proportional to the impedance of the speaker. Determine the power that would be delivered to a 4 ohm loudspeaker when connected to an amplifier that delivers 20V.

From the information given, and using *P*, *V* and *Z* to represent power, voltage and impedance, we can obtain the following relationships:

$P \propto V^2$ (power, *P*, is proportional to the square of the voltage, *V*)

and

$P \propto \frac{1}{Z}$ (power, *P*, is inversely proportional to the impedance, *Z*)

We can combine these two relationships to obtain an *equation*:

$$P = \frac{V^2}{Z}$$

Now we know that V = 20V and Z = 4 ohm. Replacing the symbols that we have been using by the values that we know gives:

$$P = \frac{V^2}{Z} = \frac{20^2}{4} = \frac{400}{4} = \mathbf{100W}$$

Problems 3

1. The density of a body is directly proportional to its mass and inversely proportional to its volume. Use the symbols, *ρ*, *m*, and *V*, to write down an expression for density in terms of mass and volume.

2. Use the relationship in 1. to determine the density of a block of metal alloy if it has a mass of 1.25kg and a volume of 0.05 m3.

Laws of indices

When simplifying calculations involving indices, certain basic rules or laws can be applied, called the *laws of indices*. These are listed below:

- when multiplying two or more numbers having the same base, the indices are added. Thus $2^2 \times 2^4 = 2^{2+4} = 2^6$.
- when a number is divided by a number having the same base, the indices are subtracted. Thus, $2^5/2^2 = 2^{5-2} = 2^3$.
- when a number which is raised to a power is raised to a further power, the indices are multiplied. Thus $(2^5)^2 = 2^{5\times 2} = 2^{10}$.
  when a number has an index of 0, its value is 1. Thus $2^0 = 1$.
- when a number is raised to a negative power, the number is the reciprocal of that number raised to a positive power. Thus $2^{-4} = 1/2^4$. Similarly, $1/2^{-3} = 2^3$.
- when a number is raised to a fractional power the denominator of the fraction is the root of the number and the numerator is the power. Thus $4^{3/4} = \sqrt[4]{4^2} = (2)^2 = 4$ and $25^{1/2} = \sqrt{25^1} = \pm 5$

Problems 4

Simplify each of the following expressions:

1. 3.17^2

2. 0.45^{-1}

3. $2.5^2 + 4.2^{-1}$

Standard form

A number written with one digit to the left of the decimal point and multiplied by 10 raised to some power is said to be written in *standard form*. Thus: 1234 is written as 1.234×10^3 in standard form, and 0.0456 is written as 4.56×10^{-2} in standard form.

When a number is written in standard form, the first factor is called the *mantissa* and the second factor is called the *exponent*. Thus, the number 6.8×10^3 has a mantissa of 6.8 and an exponent of 10^3.

Numbers having the same exponent can be added or subtracted in standard form by adding or subtracting the mantissae and keeping the exponent the same. Thus:

$2.3 \times 10^4 + 3.7 \times 10^4 = (2.3 + 3.7) \times 10^4 = 6.0 \times 10^4$,

and

$$5.7 \times 10^{-2} - 4.6 \times 10^{-2} = (5.7 - 4.6) \times 10^{-2}$$
$$= 1.1 \times 10^{-2}$$

When adding or subtracting numbers it is quite acceptable to express one of the numbers in non-standard form, so that both numbers have the same exponent. This makes things much easier as the following example shows:

$$2.3 \times 10^4 + 3.7 \times 10^3 = 2.3 \times 10^4 + 0.37 \times 10^4$$
$$= (2.3 + 0.37) \times 10^4$$
$$= 2.67 \times 10^4$$

Alternatively,

$2.3 \times 10^4 + 3.7 \times 10^3 = 23000 + 3700 = 26700 = 2.67 \times 10^4$

The laws of indices are used when multiplying or dividing numbers given in standard form. For example,

$$(22.5 \times 10^3) \times (5 \times 10^2) = (2.5 \times 5) \times (10^{3+2})$$
$$= 12.5 \times 10^5 \text{ or } 1.25 \times 10^6$$

Example 11

The period of a radio wave, *t*, is the reciprocal of its frequency, *f*. Thus $t = f^{-1} = \frac{1}{f}$

Calculate the period of a radio frequency signal having a frequency of 2.5 MHz.

Now f = 2.5 MHz.

Expressing this in standard form gives

$f = 2.5 \times 10^6$ Hz.

Now, since $t = f^{-1} = \frac{1}{2.5\times10^6} = \frac{10^{-6}}{2.5} = \frac{1}{2.5}\times10^{-6} = 0.4\times10^{-6} =$ $\mathbf{4 \times 10^{-7}}$ **s**